10.3 Geometric Sequences

Geometric Sequence: This sequence has a common ratio (r), or a value that is multiplied by one term to get the next term.

Ex: 2, -6, 18, -54, 162 ... $r = -3$ Ex: $1, \frac{x}{3}, \frac{x^2}{9}, \frac{x^3}{27}, \ldots$ $r = \frac{x}{3}$

Ex: 99, 33, 11, $\frac{11}{3}$, ... $r = \frac{1}{3}$ Ex: $1, \frac{-x}{3}, \frac{x^2}{9}, \frac{-x^3}{27}, \ldots$ $r = -\frac{x}{3}$

Ex: $\frac{2}{25}, \frac{2}{5}, 2, 10, 50, \ldots$ $r = 5$ Ex: $10, 10^{2x-1}, 10^{4x-3}, 10^{6x-5}, \ldots$ $r = \frac{10^{2x-1}}{10^1} = 10^{2x-2}$

Ex: 4, -6, 9, -13.5, ... $r = -1.5 = -\frac{3}{2}$ Ex: $1, -\sqrt{3}, 3, -3\sqrt{3}, \ldots$ $r = -\sqrt{3}$

Let's derive a formula for finding the **nth term** of a geometric sequence by looking at an example.

Ex: 2, 6, 18, 54, 162, ... r = 3 talk about the formula in reference to the first term

$2\cdot1, 2\cdot3, 2\cdot9, 2\cdot27, 2\cdot81, \ldots$
$2\cdot3^0, 2\cdot3^1, 2\cdot3^2, 2\cdot3^3, 2\cdot3^4, \ldots$
it looks like the formula will be $a_n = 2\cdot3^{n-1}$

In general, $\boxed{a_n = a_1 \cdot r^{n-1}}$ **(Memorize this!)**

Now, find a_n for the first 8 examples.

1. $a_n = 2\cdot(-3)^{n-1}$
2. $a_n = 99(\frac{1}{3})^{n-1}$
3. $a_n = \frac{2}{25}(5)^{n-1}$
4. $a_n = 4(-\frac{3}{2})^{n-1}$
5. $a_n = 1(\frac{x}{3})^{n-1}$
6. $a_n = 1(-\frac{x}{3})^{n-1}$
7. $a_n = 10(10^{2x-2})^{n-1}$
8. $a_n = 1(-\sqrt{3})^{n-1}$

If you are asked to find a later term, find a_n and plug in your specific value for n.

Ex: Find the 9th term of the geometric sequence 99, 33, 11, $\frac{11}{3}$, ... $a_1 = 99$ $r = \frac{1}{3}$ $n = 9$

$a_9 = 99(\frac{1}{3})^8$
$a_9 = \frac{99}{6561} = \boxed{\frac{11}{729}}$

Ex: Find the 6th term of geometric sequence $1, \frac{-x}{3}, \frac{x^2}{9}, \frac{-x^3}{27}, \ldots$ $a_1 = 1$ $r = -\frac{x}{3}$ $n = 6$

$a_6 = 1(-\frac{x}{3})^5 = \boxed{\frac{-x^5}{243}}$

Ex: Find the 12th term of the geometric sequence whose first two terms are 4 and 12. $a_1 = 4$ $r = 3$ $n = 12$

$a_{12} = 4(3)^{11}$
$= 4(177147)$
$= \boxed{708588}$

Sometimes you need to find r, or a_1, or another term based upon two separated terms.

Ex: Find all possible values of r for a geometric sequence given $a_3 = 3$ and $a_6 = 81$ $\frac{81}{3} = 27 \Rightarrow \sqrt[3]{27} = 3$ $\boxed{r = 3}$

Ex: Find all possible values of r for a geometric sequence given $a_7 = 5$ and $a_9 = 55$

$\frac{55}{5} = 11 \Rightarrow \sqrt[2]{11} = \boxed{\pm\sqrt{11}}$

Ex: The third term of a geometric sequence is 5, and the sixth term is -40. Find the 8th term.

$-\frac{40}{5} = -8 \Rightarrow \sqrt[3]{-8} = -2 \quad r = -2$

$a_3 = a_1(-2)^{3-1}$

$5 = a_1(4) \Rightarrow a_1 = \frac{5}{4} \quad a_n = \frac{5}{4}(-2)^{n-1} \quad a_8 = \frac{5}{4}(-2)^7 = \boxed{-160}$

<u>**Sums**</u>: The sum of the first n terms is, $S_n = a_1 \frac{1-r^n}{1-r}$

Ex: Find the sum: $\sum_{k=1}^{8} 2 \cdot 3^k$ $a_1 = 2 \cdot 3^1 = 6$ $r = 3$ $S_8 = 6\,\frac{1-3^8}{1-3} = 6\,\frac{-6560}{-2} = \boxed{19{,}680}$

Ex: Find the sum: $\sum_{k=1}^{10} (-2)^k$ $a_1 = (-2)^1 = -2$ $r = -2$ $S_{10} = -2\,\frac{1-(-2)^{10}}{1-(-2)} = -2\,\frac{-1023}{3} = \boxed{682}$

<u>**Infinite Geometric Sequence**</u>: The sum starts with the first term and keeps on going!

If $|r| < 1$, the the sum is $S = \frac{a_1}{1-r}$

Ex: Find the sum of the infinite geometric series: $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$ $r = \frac{1}{3}$ $S = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 2 \cdot \frac{3}{2} = \boxed{3}$

Ex: Find the sum of the infinite geometric series: $200 - 100 + 50 - 25 + \ldots$ $r = -\frac{1}{2}$ $S = \frac{200}{1+\frac{1}{2}} = 200 \cdot \frac{2}{3} = \boxed{\frac{400}{3}}$

Ex: Find the sum of the infinite geometric series: $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \ldots$ $r = \frac{3}{2}$

$|r|$ must be less than 1
not possible to find S

Ex: Find the sum of the infinite geometric series: $1.5 + 0.015 + 0.00015 + \ldots$ $r = .01$

$S = \frac{1.5}{1-.01} = \frac{1.5}{.99} = \frac{150}{99} = \boxed{\frac{50}{33}}$

You can also use and infinite geometric series to find the rational representation (fraction) of a repeating decimal.

Ex: Find the rational number represented by the repeating decimal: $0.\overline{3}$

$a_1 = .3 \quad r = .1$

$a_2 = .03$

$a_3 = .003$

$S = \frac{.3}{1-.1} = \frac{.3}{.9} = \frac{3}{9} = \boxed{\frac{1}{3}}$

Ex: Find the rational number represented by the repeating decimal: $0.\overline{73}$

$a_1 = .73 \quad r = .01$

$a_2 = .0073$

$a_3 = .000073$

$S = \frac{.73}{1-.01} = \frac{.73}{.99} = \boxed{\frac{73}{99}}$

Ex: Find the rational number represented by the repeating decimal: $15.\overline{2}$

$a_1 = .2 \quad r = .1$

$a_2 = .02$

$a_3 = .002$

$S = \frac{.2}{1-.1} = \frac{.2}{.9} = \frac{2}{9} \quad 15 + \frac{2}{9} = \boxed{\frac{137}{9}}$

Ex: Find the rational number represented by the repeating decimal: $2.4\overline{17}$

$a_1 = .017 \quad r = .01$

$a_2 = .00017$

$a_3 = .0000017$

$S = \frac{.017}{1-.01} = \frac{.017}{.99} = \frac{17}{990} \quad 2.4 + \frac{17}{990} = \boxed{\frac{2393}{990}}$

Applications

Ex: The yearly depreciation of a certain machine is 25% of it's value at the beginning of the year. If the original cost of the machine is \$5000, what is it's value in 7 years?

$r = .75$

$a_1 = 5000$

$a_n = 5000(.25)^{n-1}$

$a_7 = 5000(.75)^{7-1}$

$= 5000(.75)^6$

$= \boxed{\$889.89}$

Ex: A rubber ball is dropped from a height of 60 ft. If it rebounds approximately one-half the distance after each fall, use an infinite geometric series to approximate the total distance the ball travels.

↓60ft ↑30ft ↓30ft ↑15ft ↓15ft

going down: 60, 30, 15, ...

going up: 30, 15, ...

$a_n = 60(\frac{1}{2})^{n-1} \rightarrow S = 60 + 60 = 120$

$a_n = 30(\frac{1}{2})^{n-1} \rightarrow S = \frac{30}{1-\frac{1}{2}} = \frac{30}{\frac{1}{2}} = 30(2) = 60$

$\boxed{\text{total } 180\text{ft}}$