Trig 8.6 DeMoivre's Theorem & n^{th} Root of Complex Numbers

DeMoivre's Theorem

given $z = r \operatorname{cis} \theta$

then $z^n = r^n \operatorname{cis}(n \cdot \theta)$

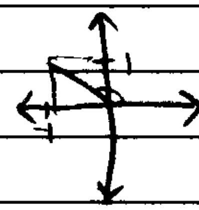
n^{th} Root of $\mathbb{C}$ Number

given $z = r \operatorname{cis} \theta$

$\sqrt[n]{z} = n$ roots

Use DeMoivre's Theorem to find

1) z^8 where $z = (-1 + i)$



$r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\tan \theta_r = \left|\frac{b}{a}\right| = \left|\frac{1}{-1}\right| = 1 \quad \theta_r = 45° \quad \theta = \frac{3\pi}{4}$

$\sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)$

$z^n = r^n \operatorname{cis}(n\theta)$

$z^8 = (\sqrt{2})^8 \operatorname{cis}\left(8 \cdot \frac{3\pi}{4}\right)$

$= \left((\sqrt{2})^2\right)^4 \operatorname{cis}\left(\cancel{8}^{2} \cdot \frac{3\pi}{\cancel{4}}\right)$

$= 16 \operatorname{cis}(6\pi)$

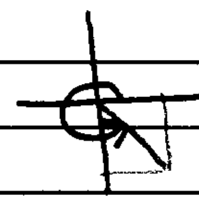
$= 16 \cdot 1 = \boxed{16}$

$\operatorname{cis}(6\pi) = \cos 6\pi + i \sin 6\pi$

$= 1 + i \cdot 0$

$= 1$

② z^3 where $z = (1 - \sqrt{3}\, i)$



$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$

$\tan \theta_r = \sqrt{3} \quad \theta_r = \frac{\pi}{3} \quad \theta = \frac{5\pi}{3}$

$2 \operatorname{cis}\left(\frac{5\pi}{3}\right)$

$z^3 = 2^3 \operatorname{cis}\left(3 \cdot \frac{5\pi}{3}\right)$

$= 8 \operatorname{cis}(5\pi)$

$= 8(\cos 5\pi + i \sin 5\pi)$

$= 8(\cos(\cancel{2 \cdot 2\pi} + \pi) + i \sin(\cancel{2 \cdot 2\pi} + \pi))$

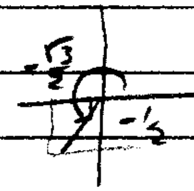
$= 8(\cos \pi + i \sin \pi)$

$= 8(-1 + i \cdot 0)$

$= \boxed{-8}$

Trig 8,6

(3) z^{20} where $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$



$r = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$

$\tan\theta_r = \left|\frac{-1/2}{-\sqrt{3}/2}\right| = \frac{\sqrt{3}}{3} \quad \theta_r = \frac{\pi}{6} \quad \theta = \frac{7\pi}{6}$

$1 \operatorname{cis}\left(\frac{7\pi}{6}\right)$

$z^{20} = 1^{20} \operatorname{cis}\left(20 \cdot \frac{7\pi}{6}\right)$

$= \operatorname{cis}\left(10 \frac{7\pi}{3}\right)$

$= \operatorname{cis}\left(\frac{70\pi}{3}\right)$

$= \operatorname{cis}\left(23 \frac{\pi}{3}\right)$

$= \operatorname{cis}\left(\cancel{22}\pi + \pi + \frac{\pi}{3}\right)$

$= \operatorname{cis}\left(\frac{4\pi}{3}\right)$

$= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$? $-i\frac{\sqrt{3}}{2}$?

n^{th} root of $z = r \operatorname{cis}\theta$

$\omega_1 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right)$

$\omega_2 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi \cdot 1}{n}\right)$

$\omega_3 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi \cdot 2}{n}\right)$

$\omega_4 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi \cdot 3}{n}\right)$

$\vdots$

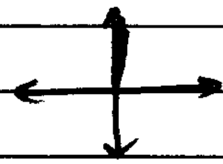
$\omega_5 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi(n-1)}{n}\right)$

Trig 8.6

Find $z = 9i \quad \sqrt{z} = ?$

$z = 9i = 9 \operatorname{cis} \pi/2$

$r = \sqrt{0^2 + 9^2} = \sqrt{9^2} = 9$



$\theta = \pi/2$

$9 \operatorname{cis} \frac{\pi}{2}$

$n = 2$

$\omega_1 = \sqrt{r} \operatorname{cis}\left(\frac{\theta}{n}\right)$

$= \sqrt{9} \operatorname{cis}\left(\frac{\pi/2}{2}\right)$

$= 3 \operatorname{cis} \pi/4$

$= 3\left(\cos \pi/4 + i \sin \frac{\pi}{4}\right)$

$= 3\left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right)$

$= \frac{3\sqrt{2}}{2} + i \frac{3\sqrt{2}}{2}$

$\omega_2 = \sqrt{r} \operatorname{cis}\left(\frac{\theta + 2\pi}{n}\right)$

$= \sqrt{9} \operatorname{cis}\left(\frac{\pi/2 + 2\pi}{2}\right)$

$= 3 \operatorname{cis} \frac{5\pi}{4}$

$= 3\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$

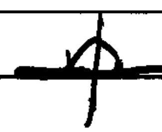
$= 3(-\cos\theta_r + i(-\sin\theta_r))$

$= 3(-\cos \pi/4 - i(\sin \pi/4))$

$= -\frac{3\sqrt{2}}{2} - i \frac{3\sqrt{2}}{2}$

Find $\sqrt[6]{z}$ with $z = -1$

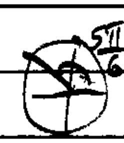
$r = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$



$\theta = \pi$

$1 \operatorname{cis} \pi$

$\operatorname{cis} \pi$



$n = 6$

$\omega_1 = \sqrt[6]{r} \operatorname{cis}\left(\frac{\theta}{n}\right)$

$= \sqrt[6]{1} \operatorname{cis} \pi/6$

$= 1 \operatorname{cis} \pi/6$

$= \operatorname{cis} \frac{\pi}{6}$

$= \cos \pi/6 + i \sin \pi/6$

$= \sqrt{3}/2 + i \, 1/2$

$\omega_2 = \operatorname{cis}\left(\frac{\pi + 2\pi}{6}\right)$

$= \operatorname{cis}\left(\frac{\pi}{2}\right)$

$= \cos \pi/2 + i \sin \frac{\pi}{2}$

$= 1 + 1i$

$\omega_3 = \operatorname{cis}\left(\frac{\pi + 4\pi}{6}\right)$

$= \operatorname{cis}\left(\frac{5\pi}{6}\right)$

$= \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}$

$= (-\cos\theta_r) + i \sin\theta_r$

$= -\cos \pi/6 + i \sin \pi/6$

$= -\frac{\sqrt{3}}{2} + i \frac{1}{2}$