

Trig 8.6 DeMoivre's Theorem & n^{th} Root of Complex Numbers

DeMoivre's Theorem

Given $z = r \operatorname{cis} \theta$
 then $z^n = r^n \operatorname{cis}(n \cdot \theta)$

n^{th} Root of \mathbb{C} Number

Given $z = r \operatorname{cis} \theta$
 $\sqrt[n]{z} = n \text{ roots}$

Use DeMoivre's Theorem to find

1) z^8 where $z = (-1 + i)$

$$r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\tan \theta_r = \left| \frac{y}{x} \right| = \left| \frac{1}{-1} \right| = 1 \quad \theta_r = 45^\circ \quad \theta = \frac{3\pi}{4}$$

$$\sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} \right)$$

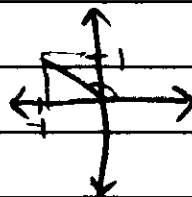
$$z^n = r^n \operatorname{cis}(n\theta)$$

$$z^8 = (\sqrt{2})^8 \operatorname{cis} \left(8 \cdot \frac{3\pi}{4} \right)$$

$$= ((\sqrt{2})^2)^4 \operatorname{cis} \left(8 \cdot \frac{3\pi}{4} \right)$$

$$= 16 \operatorname{cis}(6\pi)$$

$$= 16 \cdot 1 = \boxed{16}$$



$$\operatorname{cis}(6\pi) = \cos 6\pi + i \sin 6\pi$$

$$= 1 + i \cdot 0$$

$$= 1$$

2) z^3 where $z = (1 - \sqrt{3}i)$

$$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\tan \theta_r = \sqrt{3} \quad \theta_r = \frac{\pi}{3} \quad \theta = \frac{5\pi}{3}$$

$$2 \operatorname{cis} \left(\frac{5\pi}{3} \right)$$

$$z^3 = 2^3 \operatorname{cis} \left(3 \cdot \frac{5\pi}{3} \right)$$

$$= 8 \operatorname{cis}(5\pi)$$

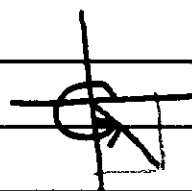
$$= 8(\cos 5\pi + i \sin 5\pi)$$

$$= 8(\cos(2 \cdot 2\pi + \pi) + i \sin(2 \cdot 2\pi + \pi))$$

$$= 8(\cos \pi + i \sin \pi)$$

$$= 8(-1 + i \cdot 0)$$

$$= \boxed{-8}$$



Trig 8.6

③ z^{20} where $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$$r = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

$$\tan \theta_r = \left| \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} \right| = \frac{1}{\sqrt{3}} \quad \theta_r = \frac{\pi}{6} \quad \theta = \frac{7\pi}{6}$$

$$1 \operatorname{cis}\left(\frac{7\pi}{6}\right)$$

$$z^{20} = 1^{20} \operatorname{cis}\left(20 \cdot \frac{7\pi}{6}\right)$$

$$= \operatorname{cis}\left(10 \frac{7\pi}{3}\right)$$

$$= \operatorname{cis}\left(\frac{70\pi}{3}\right)$$

$$= \operatorname{cis}\left(23 \frac{\pi}{3}\right)$$

$$= \operatorname{cis}\left(\cancel{22\pi} + \pi + \frac{\pi}{3}\right)$$

$$= \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

$$= \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$$

$$= -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$$\leftarrow ? -i \frac{\sqrt{3}}{2} ?$$

n^{th} root of $z = r \operatorname{cis} \theta$

$$w_1 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right)$$

$$w_2 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi \cdot 1}{n}\right)$$

$$w_3 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi \cdot 2}{n}\right)$$

$$w_4 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi \cdot 3}{n}\right)$$

\vdots

$$w_5 = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 2\pi(n-1)}{n}\right)$$

Trig 8.6

Find $z = 9i$ $\sqrt{z} = ?$

$$z = 9i = 9 \operatorname{cis} \frac{\pi}{2}$$

$$r = \sqrt{0^2 + 9^2} = \sqrt{9^2} = 9$$

$$\theta = \frac{\pi}{2}$$

$$9 \operatorname{cis} \frac{\pi}{2}$$

$$n = 2$$

$$\omega_1 = \sqrt{r} \operatorname{cis} \left(\frac{\theta}{n} \right)$$

$$= \sqrt{9} \operatorname{cis} \left(\frac{\pi/2}{2} \right)$$

$$= 3 \operatorname{cis} \frac{\pi}{4}$$

$$= 3 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$= 3 \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)$$

$$= \frac{3\sqrt{2}}{2} + i \frac{3\sqrt{2}}{2}$$

$$\omega_2 = \sqrt{r} \operatorname{cis} \left(\frac{\theta + 2\pi}{n} \right)$$

$$= \sqrt{9} \operatorname{cis} \left(\frac{\pi/2 + 2\pi}{2} \right)$$

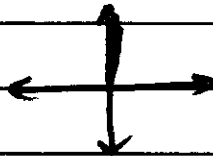
$$= 3 \operatorname{cis} \frac{5\pi}{4}$$

$$= 3 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right)$$

$$= 3 \left(-\cos \theta_r + i (-\sin \theta_r) \right)$$

$$= 3 \left(-\cos \frac{\pi}{4} - i \left(\sin \frac{\pi}{4} \right) \right)$$

$$= -\frac{3\sqrt{2}}{2} - i \frac{3\sqrt{2}}{2}$$



And $\sqrt[6]{z}$ with $z = -1$

$$r = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$$

$$\theta = \pi$$

$$1 \operatorname{cis} \pi$$

$$\operatorname{cis} \pi$$

$$n = 6$$

$$\omega_1 = \sqrt[6]{r} \operatorname{cis} \left(\frac{\theta}{n} \right)$$

$$= \sqrt[6]{1} \operatorname{cis} \frac{\pi}{6}$$

$$= 1 \operatorname{cis} \frac{\pi}{6}$$

$$= \operatorname{cis} \frac{\pi}{6}$$

$$= \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} + i \frac{1}{2}$$

$$\omega_2 = \operatorname{cis} \left(\frac{\pi + 2\pi}{6} \right)$$

$$= \operatorname{cis} \left(\frac{\pi}{2} \right)$$

$$= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$= 1 + i$$

$$\omega_3 = \operatorname{cis} \left(\frac{\pi + 4\pi}{6} \right)$$

$$= \operatorname{cis} \left(\frac{5\pi}{6} \right)$$

$$= \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}$$

$$= (-\cos \theta_r) + i \sin \theta_r$$

$$= -\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} + i \frac{1}{2}$$

